NonCommutative Rings and their Applications, VIII

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Hopf algebras versus Hopf heaps

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A Hopf algebra is an algebraic system $(H, \Delta, \varepsilon, \cdot, 1, S)$ consisting of a vector space H over a field $\mathbb{F}$ and linear maps

$$\Delta\colon H \to H \otimes H, \quad x \mapsto \sum x_1 \otimes x_2$$

$$\varepsilon\colon H \to \mathbb{F}$$

$$S\colon H \to H$$

such that

(H, Δ, ε) is a coalgebra

$$\sum \Delta(x_1) \otimes x_2 = \sum x_1 \otimes \Delta(x_2) = \sum x_1 \otimes x_2 \otimes x_3$$

$$\sum \varepsilon(x_1)x_2 = \sum x_1 \varepsilon(x_2) = x$$

$(H, \cdot, 1)$ is an algebra

the comultiplication and counit are algebra homomorphisms

$$\sum (xy)_1 \otimes (xy)_2 = \Delta(xy) = \Delta(x)\Delta(y) = \sum x_1 y_1 \otimes x_2 y_2$$

$$\Delta(1_H) = 1_H \otimes 1_H$$

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$$

$$\varepsilon(1_H) = 1_{\mathbb{F}}$$

S is an antipode

$$\sum S(x_1)x_2 = \sum x_1 S(x_2) = \varepsilon(x)1_H,$$

where $x, y \in H$.

A homomorphism of Hopf algebras $\varphi\colon (H, \Delta, \varepsilon, \cdot, 1, S_H) \to (K, \Delta, \varepsilon, \cdot, 1, S_K)$ is

a coalgebra homomorphism $\varphi\colon (H, \Delta, \varepsilon) \to (K, \Delta, \varepsilon)$

$$\Delta(\varphi(x)) = \sum \varphi(x)_1 \otimes \varphi(x)_2 = \sum \varphi(x_1) \otimes \varphi(x_2)$$

$$\varepsilon(\varphi(x)) = \varepsilon(x)$$

an algebra homomorphism $\varphi\colon (H, \cdot, 1) \to (K, \cdot, 1)$

respecting the antipode

$$\varphi(S_H(x)) = S_K(\varphi(x)),$$

where $x \in H$.




Tomasz Brzeziński, MH, *Translation Hopf algebras and Hopf heaps,*

arXiv: 2303.13154v1

A Hopf heap is an algebraic system $(C, \Delta, \varepsilon, [-,-,-])$ consisting of

a vector space C over a field $\mathbb{F}$ and linear maps

$$\Delta\colon C \to C \otimes C, \quad x \mapsto \sum x_1 \otimes x_2$$

$$\varepsilon\colon C \to \mathbb{F}$$

$$[-,-,-]\colon C \otimes C^{co} \otimes C \to C, \quad x \otimes y \otimes z \mapsto [x, y, z]$$

such that



(C, Δ, ε) is a coalgebra

$$\sum \Delta(x_1) \otimes x_2 = \sum x_1 \otimes \Delta(x_2) = \sum x_1 \otimes x_2 \otimes x_3$$

$$\sum \varepsilon(x_1) x_2 = \sum x_1 \varepsilon(x_2) = x$$

$[-,-,-]$ is a coalgebra homomorphism

$$\Delta([x,y,z]) = \sum [x,y,z]_1 \otimes [x,y,z]_2 = \sum [x_1, y_2, z_1] \otimes [x_2, y_1, z_2]$$

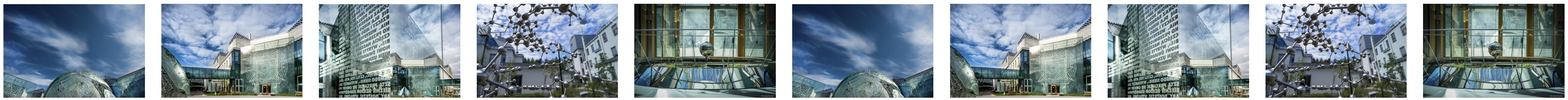
$$\varepsilon([x,y,z]) = \varepsilon(x)\varepsilon(y)\varepsilon(z)$$

satisfying

the heap associativity $\quad [[x,y,z],t,u] = [x,y,[z,t,u]] \neq [x,[y,z,t],u]$

Mal'cev identities $\quad \sum [x_1, x_2, y] = \sum [y, x_1, x_2] = \varepsilon(x) y \neq \sum [x_1, y, x_2]$,

where $x, y, z, t, u \in C$.



A homomorphism of Hopf heaps $\varphi\colon (C, \Delta, \varepsilon, [-,-,-,]) \to (D, \Delta, \varepsilon, [-,-,-,])$

is a coalgebra homomorphism $\varphi\colon (C, \Delta, \varepsilon) \to (D, \Delta, \varepsilon)$

$$\Delta(\varphi(x)) = \sum \varphi(x)_1 \otimes \varphi(x)_2 = \sum \varphi(x_1) \otimes \varphi(x_2)$$

$$\varepsilon(\varphi(x)) = \varepsilon(x)$$

respecting the heap operation

$$\varphi([x, y, z]) = [\varphi(x), \varphi(y), \varphi(z)],$$

where $x, y, z \in C$.

A Grunspan map for a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

is a coalgebra homomorphism $\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$ such that

$$[[x, y, \theta(z)], t, u] = [x, [t, z, y], u],$$

where $x, y, z, t, u \in C$.

If it exists, the Grunspan map for a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

is given by the formula

$$\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon), \quad x \mapsto \sum [x_1, [x_4, x_3, x_2], x_5],$$

where $x \in C$, and thus necessarily is unique.




The Grunspan map commutes with every homomorphism of Hopf heaps,

that is, if $\varphi\colon (C,\Delta,\varepsilon,[-,-,-]) \to (D,\Delta,\varepsilon,[-,-,-])$

is a homomorphism of Hopf heaps with respective Grunspan maps

$\theta_C\colon (C,\Delta,\varepsilon) \to (C,\Delta,\varepsilon)$ and $\theta_D\colon (D,\Delta,\varepsilon) \to (D,\Delta,\varepsilon)$,

then $\quad \varphi\theta_C = \theta_D\varphi.$

Indeed, for any $x \in C$,

$$\varphi\theta_C(x) = \varphi(\sum[x_1,[x_4,x_3,x_2],x_5)] = \sum[\varphi(x_1),[\varphi(x_4),\varphi(x_3),\varphi(x_2)],\varphi(x_1)] =$$

$$= \sum[\varphi(x)_1,[\varphi(x)_4,\varphi(x)_3,\varphi(x)_2],\varphi(x)_5] = \theta_D\varphi(x).$$

Theorem. Every Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

admits the Grunspan map $\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$.

See [Tomasz Brzeziński, MH, *Translation Hopf algebras and Hopf heaps,*

arXiv: 2303.13154v1, Corollary 3.9]. □

Theorem. Given a Hopf algebra $(H, \Delta, \varepsilon, \cdot, 1, S)$, let

$$[-,-,-]_\bullet \colon H \otimes H^{co} \otimes H \to H, \quad [x,y,z]_\bullet = x \cdot S(y) \cdot z$$

$$\theta \colon H \to H, \quad \theta(x) = S^2(x),$$

where $x, y, z \in H$. Then

(a) $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$ is a Hopf heap

with the Grunspan map $\theta \colon (H, \Delta, \varepsilon) \to (H, \Delta, \varepsilon)$.

Indeed, for any $x, y, z, t, u \in H$,

$$\begin{aligned}
\Delta([x,y,z]_\bullet) &= \Delta(x \cdot S(y) \cdot z) = \Delta(x) \cdot \Delta(S(y)) \cdot \Delta(z) = \\
&= \sum (x_1 \otimes x_2) \cdot (S(y)_1 \otimes S(y)_2) \cdot (z_1 \otimes z_2) = \\
&= \sum (x_1 \otimes x_2) \cdot (S(y_2) \otimes S(y_1)) \cdot (z_1 \otimes z_2) = \\
&= \sum x_1 \cdot S(y_2) \cdot z_1 \otimes x_2 \cdot S(y_1) \cdot z_2 = \sum [x_1, y_2, z_1]_\bullet \otimes [x_2, y_1, z_2]_\bullet
\end{aligned}$$

$$\varepsilon([x,y,z]_\bullet) = \varepsilon(x \cdot S(y) \cdot z) = \varepsilon(x)\varepsilon(S(y))\varepsilon(z) = \varepsilon(x)\varepsilon(y)\varepsilon(z)$$

$$[[x,y,z]_\bullet, t, u]_\bullet = (x \cdot S(y) \cdot z) \cdot S(t) \cdot u = x \cdot S(y) \cdot (z \cdot S(t) \cdot u) = [x, y, [z, y, u]_\bullet]_\bullet$$

$$\sum [x_1, x_2, y]_\bullet = \sum x_1 \cdot S(x_2) \cdot y = \varepsilon(x) y = \sum y \cdot S(x_1) \cdot x_2 = \sum [y, x_1, x_2]_\bullet$$

$$[[x, y, \theta(z)]_\bullet, t, u]_\bullet = x \cdot S(y) \cdot S^2(z) \cdot S(t) \cdot u = x \cdot S(t \cdot S(z) \cdot y) \cdot u = [x, [t, z, y]_\bullet, u]_\bullet$$

(b) Every homomorphism of Hopf algebras

$$\varphi\colon (H, \Delta, \varepsilon, \cdot, 1, S_H) \to (K, \Delta, \varepsilon, S_K)$$

is a homomorphism of associated Hopf heaps

$$\varphi\colon (H, \Delta, \varepsilon, [-,-,-]_\bullet) \to (K, \Delta, \varepsilon, [-,-,-]_\bullet).$$

Indeed, for any $x, y, z \in H$,

$$\varphi([x,y,z]_\bullet) = \varphi(x \cdot S_H(y) \cdot z) = \varphi(x) \cdot \varphi(S_H(y)) \cdot \varphi(z) =$$

$$= \varphi(x) \cdot S_K(\varphi(y)) \cdot \varphi(z) = [\varphi(x), \varphi(y), \varphi(z)]_\bullet \qquad \square$$



Theorem. Given a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

and a group-like element $e \in G(C)$, let

$$\cdot_e \colon C \otimes C \to C, \quad x \cdot_e y = [x, e, y]$$

$$S_e \colon C \to C, \quad S_e(x) = [e, x, e]$$

where $x, y \in C$. Then

(a) $(C, \Delta, \varepsilon, \cdot_e, e, S_e)$ is a Hopf algebra denoted by $H_e(C)$.



Indeed, for any $x, y, z \in C$,

$$(x \cdot_e y) \cdot_e z = [[x, e, y], e, z] = [x, e, [y, e, z]] = x \cdot_e (y \cdot_e z)$$

$$e \cdot_e x = [e, e, x] = x = [x, e, e] = x \cdot_e e$$

$$\begin{aligned} \Delta(x \cdot_e y) = \Delta([x, e, y]) = \sum [x_1, e, y_1] \otimes [x_2, e, y_2] = \\ = \sum x_1 \cdot_e y_1 \otimes x_2 \cdot_e y_2 = \sum (x_1 \otimes x_2) \cdot_e (y_1 \otimes y_2) = \Delta(x) \cdot_e \Delta(y) \end{aligned}$$

$$\Delta(e) = e \otimes e$$

$$\varepsilon(x \cdot_e y) = \varepsilon([x, e, y]) = \varepsilon(x)\varepsilon(e)\varepsilon(y) = \varepsilon(x)\varepsilon(y)$$

$$\varepsilon(e) = 1_{\mathbb{F}}$$

$$\sum S_e(x_1) \cdot_e x_2 = \sum [[e, x_1, e], e, x_2] = \sum [e, x_1, [e, e, x_2]] = \sum [e, x_1, x_2] = \varepsilon(x)e$$

$$\sum x_1 \cdot_e S_e(x_2) = \sum [x_1, e, [e, x_2, e]] = \sum [[x_1, e, e], x_2, e] = \sum [x_1, x_2, e] = \varepsilon(x)e.$$

(b) If $\varphi\colon (C,\Delta,\varepsilon,[-,-,-]) \to (D,\Delta,\varepsilon,[-,-,-])$

is a homomorphism of Hopf heaps,

then for group-like elements $e \in G(C)$ and $f \in G(D)$, the maps

$$\widehat{\varphi}\colon (H_e(C),\Delta,\varepsilon,\cdot_e,e) \to (H_f(D),\Delta,\varepsilon,\cdot_f,f), \quad \widehat{\varphi}(x) = [\varphi(x),\varphi(e),f]$$

$$\widehat{\varphi}^\circ\colon (H_e(C),\Delta,\varepsilon,\cdot_e,e) \to (H_f(D),\Delta,\varepsilon,\cdot_f,f), \quad \widehat{\varphi}^\circ(x) = [f,\varphi(e),\varphi(x)]$$

are associated bialgebra homomorphisms.



Indeed, for any $x, y \in C$,

$$\begin{aligned}\widehat{\varphi}(x \cdot_e y) &= \widehat{\varphi}([x, e, y]) = [\varphi([x, e, y]), \varphi(e), f] = [[\varphi(x), \varphi(e), \varphi(y)], \varphi(e), f] = \\ &= [\varphi(x), \varphi(e), [\varphi(y), \varphi(e), f]] = [\varphi(x), \varphi(e), \widehat{\varphi}(y)] = \\ &= [\varphi(x), \varphi(e), [f, f, \widehat{\varphi}(y)]] = [[\varphi(x), \varphi(e), f], f, \widehat{\varphi}(y)] = [\widehat{\varphi}(x), f, \widehat{\varphi}(y)] = \widehat{\varphi}(x) \cdot_f \widehat{\varphi}(y)\end{aligned}$$

Since $\varphi\colon (C, \Delta, \varepsilon) \to (D, \Delta, \varepsilon)$ is a coalgebra map and $e \in G(C)$ is a group-like element, it follows that

$$\Delta(\varphi(e)) = \sum \varphi(e)_1 \otimes \varphi(e)_2 = \varphi(e) \otimes \varphi(e), \quad \varepsilon(\varphi(e)) = \varepsilon(e) = 1_{\mathbb{F}}.$$

This means that $\varphi(e) \in G(D)$ is a group-like element, and hence

$$\widehat{\varphi}(e) = [\varphi(e), \varphi(e), f] = f$$

$$\begin{aligned}\Delta(\widehat{\varphi}(x)) &= \sum \widehat{\varphi}(x)_1 \otimes \widehat{\varphi}(x)_2 = \\ &= \sum [\varphi(x), \varphi(e), f]_1 \otimes [\varphi(x), \varphi(e), f]_2 = \\ &= \sum [\varphi(x)_1, \varphi(e)_2, f] \otimes [\varphi(x)_2, \varphi(e)_1, f] = \\ &= \sum [\varphi(x_1), \varphi(e), f] \otimes [\varphi(x_2), \varphi(e), f] = \sum \widehat{\varphi}(x_1) \otimes \widehat{\varphi}(x_2)\end{aligned}$$

$$\varepsilon(\widehat{\varphi}(x)) = \varepsilon([\varphi(x), \varphi(e), f]) = \varepsilon(\varphi(x))\varepsilon(\varphi(e))\varepsilon(f) = \varepsilon(x).$$

(c) If both the Hopf heaps

$(C, \Delta, \varepsilon, [-,-,-])$ and $(D, \Delta, \varepsilon, [-,-,-])$

admit the Grunspan maps

$\theta_C \colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$ and $\theta_D \colon (D, \Delta, \varepsilon) \to (D, \Delta, \varepsilon)$,

then both the maps

$$\widehat{\varphi} \colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f(D), \Delta, \varepsilon, \cdot_f, f), \quad \widehat{\varphi}(x) = [\varphi(x), \varphi(e), f]$$

$$\widehat{\varphi}^{\circ} \colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f(D), \Delta, \varepsilon, \cdot_f, f), \quad \widehat{\varphi}^{\circ}(x) = [f, \varphi(e), \varphi(x)]$$

are homomorphisms of associated Hopf algebras.




Indeed, for any $x \in C$,

$$S_f(\widehat{\varphi}(x)) = [f, \widehat{\varphi}(x), f] = [f, [\varphi(x), \varphi(e), f], f] =$$
$$= [[f, f, \theta_D(\varphi(e))], \varphi(x), f] = [\theta_D(\varphi(e)), \varphi(x), f] =$$
$$= [\varphi(\theta_C(e)), \varphi(x), f] = [\varphi(e), \varphi(x), f] =$$
$$= [\varphi(e), \varphi(x), [\varphi(e), \varphi(e), f]] = [[\varphi(e), \varphi(x), \varphi(e)], \varphi(e), f] =$$
$$= [\varphi([e, x, e]), \varphi(e), f] = \widehat{\varphi}([e, x, e]) = \widehat{\varphi}(S_e(x)). \qquad \square$$



Corollary. Given a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

and group-like elements $e, f \in G(C)$, let

$(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$ and $(H_f(C), \Delta, \varepsilon, \cdot_f, f, S_f)$ be the Hopf algebras

associated to the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$.

Then the map

$$\tau_e^f\colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f(C), \Delta, \varepsilon, \cdot_f, f), \quad \tau_e^f(x) = [x, e, f]$$

is an associated bialgebra isomorphism with the inverse $(\tau_e^f)^{-1} = \tau_f^e$.

If the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

admits the Grunspan map $\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$,

then τ_e^f is an isomorphism of associated Hopf algebras.

Indeed, the map

$$\tau_e^f = \widehat{\mathrm{id}_C} \colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f, \Delta, \varepsilon, \cdot_f, f), \quad \tau_e^f(x) = [\mathrm{id}_C(x), \mathrm{id}_C(e), f] = [x, e, f]$$

is a bialgebra isomorphism. $\square$

A Hopf algebra $(H, \Delta, \varepsilon, \cdot, 1, S)$

$$\Downarrow$$

The Hopf heap $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$

associated to the Hopf algebra $(H, \Delta, \varepsilon, \cdot, 1, S)$,

where $[x, y, z]_\bullet = x \cdot S(y) \cdot z$

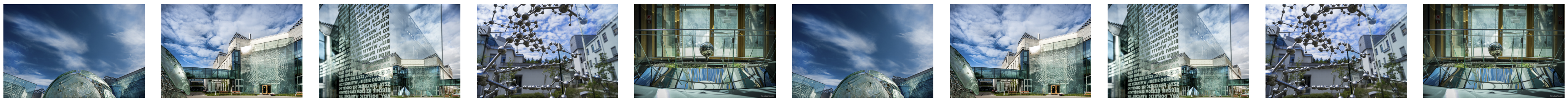
$$\Downarrow$$

$\forall e \in G(H)$, The Hopf algebra $(H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e)$

associated to the Hopf heap $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$,

where $x \cdot_e y = [x, e, y]_\bullet = x \cdot S(e) \cdot y$

$$S_e(x) = [e, x, e]_\bullet = e \cdot S(x) \cdot e$$



A Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

$$\Downarrow$$

$\forall e \in G(C)$, The Hopf algebra $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$

associated to the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$,

where $x \cdot_e y = [x, e, y]$

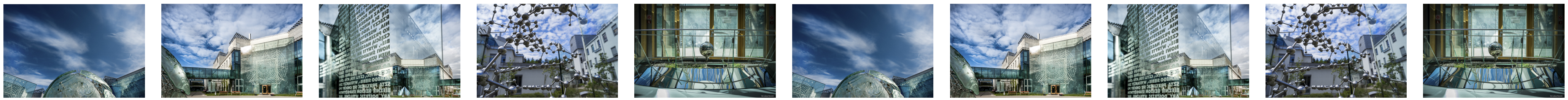
$$S_e(x) = [e, x, e]$$

$$\Downarrow$$

The Hopf heap $(H_e(C), \Delta, \varepsilon, [-,-,-]_{\bullet_e})$

associated to the Hopf algebra $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$,

where $[x, y, z]_{\bullet_e} = x \cdot_e S_e(y) \cdot_e z = [[x, e, S_e(y)], e, z] = [[x, e, [e, y, e]], e, z]$



Theorem. Given a Hopf algebra $(H, \Delta, \varepsilon, \cdot, 1, S)$

and a group-like element $e \in G(H)$, let

$(H, \Delta, \varepsilon, [-,-,-]_\bullet)$ be the Hopf heap

associated to the Hopf algebra $(H, \Delta, \varepsilon, \cdot, 1, S)$, let

$(H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e)$ be the Hopf algebra

associated to the Hopf heap $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$.

Then $(H, \Delta, \varepsilon, \cdot, 1, S) \cong (H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e)$ as Hopf algebras.

In particular $\cdot = \cdot_1$.




Indeed, let $\varphi\colon H \to H_e(H)$, $\quad \varphi(x) = x \cdot e$.

Then for any $x, y \in H$,

$$\varphi(x \cdot y) = (x \cdot y) \cdot e = (x \cdot e) \cdot e^{-1} \cdot (y \cdot e) = \varphi(x) \cdot S(e) \cdot \varphi(y) = [\varphi(x), e, \varphi(y)]_\bullet = \varphi(x) \cdot_e \varphi(y)$$

$$\varphi(1) = 1 \cdot e = e$$

$$\Delta(\varphi(x)) = \Delta(x \cdot e) = \Delta(x) \cdot \Delta(e) = \sum (x_1 \otimes x_2) \cdot (e \otimes e) = \sum x_1 \cdot e \otimes x_2 \cdot e = \sum \varphi(x_1) \otimes \varphi(x_2)$$

$$\varepsilon(\varphi(x)) = \varepsilon(x \cdot e) = \varepsilon(x)\varepsilon(e) = \varepsilon(x)$$

$$\varphi(S(x)) = S(x) \cdot e = e \cdot e^{-1} \cdot S(x) \cdot e = e \cdot S(e) \cdot S(x) \cdot e =$$

$$= e \cdot S(x \cdot e) \cdot e = e \cdot S(\varphi(x)) \cdot e = [e, \varphi(x), e]_\bullet = S_e(\varphi(x)).$$

Hence φ is an isomorphism of Hopf algebrs with the inverse

$$\varphi^{-1}\colon (H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e) \to (H, \Delta, \varepsilon, \cdot, 1, S), \quad \varphi^{-1}(x) = x \cdot e^{-1}.$$

$$x \cdot y = x \cdot 1 \cdot y = x \cdot S(1) \cdot y = [x, 1, y]_\bullet = x \cdot_1 y.$$ □




Theorem. Given a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

and a group-like element $e \in G(C)$, let

$(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$ be the Hopf algebra

associated to the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$, let

$(H_e(C), \Delta, \varepsilon, [-,-,-]_{\bullet_e})$ be the Hopf heap

associated to the Hopf algebra $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$.

Then $[-,-,-] = [-,-,-]_{\bullet_e}$.

Indeed, for any $x, y, z \in C$,

$$\begin{aligned}[x,y,z] &= [x,y,[e,e,z]] = [[x,y,e],e,z] = \\ &= [[[x,e,e],y,e],e,z] = [[x,e,[e,y,e]],e,z] = \\ &= [[x,e,S_e(y)],e,z] = x \cdot_e S_e(y) \cdot_e z = [x,y,z]_{\bullet_e}.\end{aligned}$$

□






Let $(C, \Delta, \varepsilon, [-,-,-])$ be a Hopf heap.

For any $a, b \in C$, the linear map

$$\tau_a^b \colon C \to C, \qquad \tau_a^b(x) = [x, a, b]$$

is called a right (a,b)-translation.

The space spanned by all right (a,b)-translations is denoted by $\mathsf{Tn}(C)$,

that is,

$$\mathsf{Tn}(C) = \mathbb{F}\langle \tau_a^b \mid a, b \in C\rangle.$$

Symmetrically, linear maps

$$\sigma_a^b \colon C \to C, \quad \sigma_a^b(x) = [a, b, x]$$

are called left (a,b)-translations

and the space spanned by all of them is denoted by $\widehat{\mathsf{Tn}}(C)$.

Theorem. Let $(C, \Delta, \varepsilon, [-,-,-])$ be a Hopf heap. Then

(a) The space $\mathrm{Tn}(C)$ is a bialgebra with the multiplication given by

the opposite composition, with the comultiplication

$$\Delta(\tau_a^b) = \sum \tau_{a_2}^{b_1} \otimes \tau_{a_1}^{b^2}$$

and counit

$$\varepsilon(\tau_a^b) = \varepsilon(a)\varepsilon(b).$$

(b) If the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

admits the Grunspan map $\theta\colon (C, \Delta, \varepsilon, [-,-,-]) \to (C, \Delta, \varepsilon, [-,-,-])$,

then $\mathrm{Tn}(C)$ is a Hopf algebra with the antipode

$$S(\tau_a^b) := \tau_b^{\theta(a)}.$$

(c) If $f: C \to D$ is a homomorphism of Hopf heaps, then the map

$$\mathrm{Tn}(f): \mathrm{Tn}(C) \to \mathrm{Tn}(D), \quad \mathrm{Tn}(f)(\tau_a^b) = \tau_{f(a)}^{f(b)}$$

is a bialgebra homomorphism,

hence a Hopf algebra homomorphism whenever the Grunspan map exists.

(d) The assignment

$$C \mapsto \mathrm{Tn}(C), \quad f \mapsto \mathrm{Tn}(f)$$

defines a functor from the category of Hopf heaps (with Grunspan maps)

to the category of bialgebras (resp. Hopf algebras). $\square$




Corollary. Every Hopf heap admits the Grunspan map.

Thank you very much for your attention!

Merci beaucoup pour votre attention!

La ringrazio molto per l'attenzione!

























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